

# Spectra of edge-independent random graphs

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## Abstract

Let  $G$  be a random graph on the vertex set  $\{1, 2, \dots, n\}$  such that edges in  $G$  are determined by independent random indicator variables, while the probability  $p_{ij}$  for  $\{i, j\}$  being an edge in  $G$  is not assumed to be equal. Spectra of the adjacency matrix and the normalized Laplacian matrix of  $G$  are recently studied by Oliveira and Chung-Radcliffe. Let  $A$  be the adjacency matrix of  $G$ ,  $\bar{A} = E(A)$ , and  $\Delta$  be the maximum expected degree of  $G$ . Oliveira first proved that almost surely  $\|A - \bar{A}\| = O(\sqrt{\Delta \ln n})$  provided  $\Delta \geq C \ln n$  for some constant  $C$ . Chung-Radcliffe improved the hidden constant in the error term using a new Chernoff-type inequality for random matrices. Here we prove that almost surely  $\|A - \bar{A}\| \leq (2 + o(1))\sqrt{\Delta}$  with a slightly stronger condition  $\Delta \gg \ln^4 n$ . For the Laplacian  $L$  of  $G$ , Oliveira and Chung-Radcliffe proved similar results  $\|L - \bar{L}\| = O(\sqrt{\ln n}/\sqrt{\delta})$  provided the minimum expected degree  $\delta \gg \ln n$ ; we also improve their results by removing the  $\sqrt{\ln n}$  multiplicative factor from the error term under some mild conditions. Our results naturally apply to the classic Erdős-Rényi random graphs, random graphs with given expected degree sequences, and bond percolation of general graphs.

## 1 Introduction

Given an  $n \times n$  symmetric matrix  $M$ , let  $\lambda_1(M), \lambda_2(M), \dots, \lambda_n(M)$  be the list of eigenvalues of  $M$  in the non-decreasing order. What can we say about these eigenvalues if  $M$  is a matrix associated with a random graph  $G$ ? Here  $M$  could be the adjacency matrix (denoted by  $A$ ) or the normalized Laplacian matrix (denoted by  $L$ ). Both spectra of  $A$  and  $L$  can be used to infer structures of  $G$ . For example, the spectrum of  $A$  is related to the chromatic number and the independence number. The spectrum of  $L$  is connected to the mixing-rate of random walks, the diameters, the neighborhood expansion, the Cheeger constant, the isoperimetric inequalities, the expander graphs, the quasi-random graphs. For more applications of spectra of the adjacency matrix and the Laplacian matrix, please refer to monographs [3, 9].

Spectra of adjacency matrices and normalized Laplacian matrices of random graphs were extensively investigated in the literature. For the Erdős-Rényi random graph model  $G(n, p)$ , Füredi and Komlós [15] proved that almost surely  $\lambda_n(A) = (1 + o(1))np$  and  $\max_{i \leq n-1} |\lambda_i(A)| \leq (2 + o(1))\sqrt{np(1-p)}$  provided  $np(1-p) \gg \log^6 n$ ; similar results are proved for sparse random graphs [11, 16] and general random matrices [10, 15]. Alon, Krivelevich, and Vu [1] showed that with high probability the  $s$ -th largest eigenvalue of a random

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symmetric matrix with independent random entries of absolute value at most one concentrates around its median. Chung, Lu, and Vu [4, 5] studied spectra of adjacency matrices of random power law graphs and spectra of Laplacian matrices of random graphs with given expected degree sequences. Their results on random graphs with given expected degree sequences were complemented by Coja-Oghlan [7, 8] for sparser cases. For random  $d$ -regular graphs, Friedman (in a series of papers) [12, 13, 14] proved that the second largest eigenvalue (in absolute value) of random  $d$ -regular graphs is at most  $(2 + o(1))\sqrt{d-1}$  almost surely for any  $d \geq 4$ .

In this paper, we study spectra of the adjacency matrices and the Laplacian matrices of edge-independent random graphs. Let  $G$  be an edge-independent random graph on the vertex set  $[n] := \{1, 2, \dots, n\}$ ; two vertices  $i$  and  $j$  are adjacent in  $G$  with probability  $p_{ij}$  independently. Here  $\{p_{ij}\}_{1 \leq i, j \leq n}$  are not assumed to be equal. Let  $\bar{A} := (p_{ij})_{i, j=1}^n$  be the expectation of the adjacency matrix  $A$  and  $\Delta$  be the maximum expected degree of  $G$ . Oliveira [17] proved  $\|A - \bar{A}\| = O(\sqrt{\Delta \ln n})$  provided  $\Delta \geq C \ln n$  for some constant  $C$ . Chung and Radcliffe [6] improved the hidden constant in the error term using a Chernoff-type inequality for random matrices. We manage to remove the  $\sqrt{\ln n}$ -factor from the error term with a slightly stronger assumption on  $\Delta$ . We have the following theorem.

**Theorem 1** *Consider an edge-independent random graph  $G$ . If  $\Delta \gg \ln^4 n$ , then almost surely*

$$|\lambda_i(A) - \lambda_i(\bar{A})| \leq (2 + o(1))\sqrt{\Delta}$$

for each  $1 \leq i \leq n$ .

Let  $T$  be the diagonal matrix of expected degrees. Define  $\bar{L} = I - T^{-1/2} \bar{A} T^{-1/2}$ . The matrix  $\bar{L}$  can be viewed as the “expected Laplacian” of  $G$ . Oliveira [17] and Chung-Radcliffe [6] proved theorems on  $L$  which are similar to those on the adjacency matrix  $A$ . We are able to improve their results by removing the  $\sqrt{\ln n}$ -factor from the error term with some conditions. We say that  $\bar{L}$  is well-approximated by a rank  $k$ -matrix if there is a  $k$  such that all but  $k$  eigenvalues  $\lambda_i(\bar{L})$  satisfy  $|1 - \lambda_i(\bar{L})| = o(1/\sqrt{\ln n})$ . To make the definition rigorous, let

$$\Lambda := \{\lambda_i(L) : |1 - \lambda_i(L)| \geq 1/(g(n)\sqrt{\ln n})\},$$

where  $g(n)$  is an arbitrarily slowly growing function; then we have  $k := |\Lambda|$ . We have the following theorem.

**Theorem 2** *Consider an edge-independent random graph  $G$ . Let  $\Lambda$  and  $k$  be defined above. If the minimum expected degree  $\delta$  satisfies  $\delta \gg \max\{k, \ln^4 n\}$ , then almost surely,*

$$|\lambda_i(L) - \lambda_i(\bar{L})| \leq \left(2 + \sqrt{\sum_{\lambda \in \Lambda} (1 - \lambda)^2} + o(1)\right) \frac{1}{\sqrt{\delta}}$$

for each  $1 \leq i \leq n$ .

Note  $\text{rank}(I - \bar{L}) = \text{rank}(\bar{A})$ . We have the following corollary.

**Corollary 1** *Consider an edge-independent random graph  $G$  with  $\text{rank}(\bar{A}) = k$ . If the minimum expected degree  $\delta$  satisfies  $\delta \gg \max\{k, \ln^4 n\}$ , then almost surely, we have*

$$|\lambda_i(L) - \lambda_i(\bar{L})| \leq \frac{2 + \sqrt{k} + o(1)}{\sqrt{\delta}}$$

for  $1 \leq i \leq n$ .

A special case is the random graph  $G(\mathbf{w})$  with given expected degree sequence  $\mathbf{w} = (w_1, w_2, \dots, w_n)$ , where  $v_i v_j$  is an edge with probability  $p_{ij} = \frac{w_i w_j}{\sum_{l=1}^n w_l}$ . Let  $\delta = w_{\min}$  and  $\bar{w} = \frac{\sum_{i=1}^n w_i}{n}$ . Chung-Lu-Vu [5] proved if  $\delta \gg \sqrt{\bar{w}} \ln^3 n$  then for each non-trivial eigenvalue  $\lambda_i(L)$ , we have

$$|1 - \lambda_i(L)| \leq \frac{4 + o(1)}{\sqrt{\delta}}. \quad (1)$$

Note that in this case  $I - \bar{L} = T^{-1/2} \bar{A} T^{-1/2}$ ; its  $(i, j)$ -entry is given by  $\frac{\sqrt{w_i w_j}}{\sum_{l=1}^n w_l}$ . Thus  $I - \bar{L}$  is a rank-1 matrix with non-zero eigenvalues equal 1. Hence all non-trivial eigenvalues of  $\bar{L}$  are 1. Applying Corollary 1, we get

$$|1 - \lambda_i(L)| \leq \frac{3 + o(1)}{\sqrt{\delta}}, \quad (2)$$

provided  $\delta \gg \ln^4 n$ .

In comparison to inequality (1), inequality (2) improves the constant factor with a weaker condition.

Here is another application. Let  $G$  be a host graph with vertex set  $[n]$ . The bond percolation of  $G$  (with probability  $p$ ) is a random spanning subgraph  $G_p$  of  $G$  such that for each edge  $\{i, j\}$  of  $G$ ,  $\{i, j\}$  is retained as an edge of  $G_p$  with probability  $p$  independently. The Erdős-Rényi graph  $G(n, p)$  can be viewed as the bond percolation of the complete graph  $K_n$ . We have the following theorems on the spectrum of  $G_p$  for a general graph  $G$ .

**Theorem 3** *Suppose that the maximum degree  $\Delta$  of  $G$  satisfies  $\Delta \gg \ln^4 n$ . For  $p \gg \frac{\ln^4 n}{\Delta}$ , almost surely we have*

$$|\lambda_i(A(G_p)) - p\lambda_i(A(G))| \leq (2 + o(1))\sqrt{p\Delta}.$$

**Theorem 4** *Suppose that all but  $k$  Laplacian eigenvalues  $\lambda$  of  $G$  satisfies  $|1 - \lambda| = o(\frac{1}{\sqrt{\ln n}})$ . If the minimum degree  $\delta$  of  $G$  satisfies  $\delta \gg \max\{k, \ln^4 n\}$ , then for  $p \gg \max\{\frac{k}{\delta}, \frac{\ln^4 n}{\delta}\}$ , almost surely we have*

$$|\lambda_i(L(G_p)) - \lambda_i(L(G))| \leq \left(2 + \sqrt{\sum_{i=1}^k (1 - \lambda_i)^2 + o(1)}\right) \frac{1}{\sqrt{p\delta}},$$

where  $\lambda_1, \dots, \lambda_k$  are those  $k$  Laplacian eigenvalues of  $G$  do not satisfy  $|1 - \lambda| = o(\frac{1}{\sqrt{\ln n}})$ .

The rest of the paper is organized as follows. In section 2, we will generalize Vu's result [18] on the spectral bound of a random symmetric matrix; we use it to prove theorem 1. In section 3, we will prove several lemmas for Laplacians. Finally, Theorem 2 will be proved in Section 4.

## 2 Spectral bound of random symmetric matrices

For any matrix  $M$ , the *spectral norm*  $\|M\|$  is the largest singular value of  $M$ ; i.e., we have

$$\|M\| = \sqrt{\lambda_{\max}(M^* M)}.$$

Here  $M^*$  is the conjugate transpose of  $M$  and  $\lambda_{\max}(\bullet)$  is the largest eigenvalue. When  $M$  is a symmetric matrix with non-negative entries, we have  $\|M\| = \lambda_{\max}(M)$ .

We will estimate the spectral norm of random symmetric matrices. Let us start with the following theorem proved by Vu in [18].

**Theorem 5** *There are constants  $C$  and  $C'$  such that the following holds. Let  $b_{ij}$ ,  $1 \leq i \leq j \leq n$  be independent random variables, each of which has mean 0 and variance at most  $\sigma^2$  and is bounded in absolute value by  $K$ , where  $\sigma \geq C'n^{-1/2}K \ln^2 n$ . Then almost surely*

$$\|B\| \leq 2\sigma\sqrt{n} + C(K\sigma)^{1/2}n^{1/4} \ln n. \quad (3)$$

Vu's theorem is already in a very general form; it improves Füredi-Komlós's result [15] on  $G(n, p)$ . When we consider an edge-independent random graph  $G$ , let  $A$  be the adjacency matrix of  $G$  and  $\bar{A}$  be the expectation of  $A$ . If we apply Theorem 5 to  $B := A - \bar{A}$ , we get

$$\|A - \bar{A}\| \leq 2\sigma\sqrt{n} + C(\sigma)^{1/2}n^{1/4} \ln n, \quad (4)$$

where  $\sigma = \max_{1 \leq i \leq j \leq n} \{\sqrt{p_{ij}(1-p_{ij})}\}$ . The upper bound in inequality (4) is weaker than the one in Theorem 1; this is because the uniform bounds on  $K$  and  $\sigma^2$  are too coarse.

To overcome the deficiency, we assume that  $b_{ij}$  ( $1 \leq i \leq j \leq n$ ) are independent random variables with the following properties:

- $|b_{ij}| \leq K$  for  $1 \leq i < j \leq n$ ;
- $E(b_{ij}) = 0$ , for all  $1 \leq i < j \leq n$ ;
- $\text{Var}(b_{ij}) \leq \sigma_{ij}^2$ .

If  $i > j$ , we set  $b_{ji} = b_{ij}$  and  $\sigma_{ji} = \sigma_{ij}$ . Consider a random symmetric matrix  $B = (b_{ij})_{i,j=1}^n$ . The following theorem generalizes Vu's theorem.

**Theorem 6** *There are constants  $C$  and  $C'$  such that the following holds. Let  $B$  be the random symmetric matrix defined above and  $\Delta = \max_{1 \leq i \leq n} \sum_{j=1}^n \sigma_{ij}^2$ . If  $\Delta \geq C'K^2 \ln^4 n$ , then almost surely*

$$\|B\| \leq 2\sqrt{\Delta} + C\sqrt{K}\Delta^{1/4} \ln n. \quad (5)$$

When  $\sigma_{ij} \equiv \sigma$ , we have  $\Delta = n\sigma^2$ . Thus, inequality (5) implies inequality (3). (The condition  $\Delta \geq C'K^2 \ln^4 n$  becomes  $\sigma \geq \sqrt{C'}n^{-1/2}K \ln^2 n$ .)

Replacing  $B$  by  $cB$ ,  $K$  by  $cK$ , and  $\Delta$  by  $c\Delta$ , inequality (5) is invariant under scaling. Without loss of generality, we can assume  $K = 1$  (by scaling a factor  $\frac{1}{K}$ ). We further assume that diagonal entries are zeros. Changing diagonal entries to zeros can affect the spectral norm by at most  $K$ , which is negligible in comparison to the upper bound.

We use Wigner's trace method [19]. We have

$$E(\text{Trace}(B^k)) = \sum_{i_1, i_2, \dots, i_k} E(b_{i_1 i_2} b_{i_2 i_3} \dots b_{i_{k-1} i_k} b_{i_k i_1}). \quad (6)$$

Each sequence  $w := i_1 i_2 \dots i_{k-1} i_k i_1$  is a *closed walk* of length  $k$  in the complete graph  $K_n$ . Let  $E(w)$  be the set of edges appearing in  $w$ . For each edge  $e \in E(w)$ , let  $q_e$  be the number of occurrence of the edge  $e$  in the walk  $w$ . By the independence assumption for edges, we can rewrite equation (6) as

$$E(\text{Trace}(B^k)) = \sum_w \prod_{e \in E(w)} E(b_e^{q_e}). \quad (7)$$

Here the summation is taken over all closed walks of length  $k$ . If  $q_e = 1$  for some  $e \in E(w)$ , then  $\prod_{e \in E(w)} E(b_e^{q_e}) = 0$ . Thus we need only to consider all closed walks such that each edge appears at least twice.

A closed walk  $w$  is *good* if each edge in  $E(w)$  occurs more than once. The set of all good closed walks of length  $k$  in  $K_n$  is denoted by  $\mathcal{G}(n, k)$ .

Since  $q_e \geq 2$  and  $|b_e| \leq 1$ , we have

$$|\mathbb{E}(b_e^{q_e})| \leq \mathbb{E}(b_e^2) = \text{Var}(b_e^2) \leq \sigma_e^2. \quad (8)$$

Putting equation (7) and inequality (8) together, we have

$$|\mathbb{E}(\text{Trace}(B^k))| \leq \sum_{w \in \mathcal{G}(n, k)} \prod_{e \in E(w)} \sigma_e^2. \quad (9)$$

Let  $\mathcal{G}(n, k, p)$  be the set of good closed walks in  $K_n$  of length  $k$  and with  $p$  vertices. The key of the trace method is a good estimate on  $|\mathcal{G}(n, k, p)|$ . Füredi and Komlós [15] proved

$$|\mathcal{G}(n, k, p)| \leq n(n-1) \cdots (n-p+1) \frac{1}{p} \binom{2p-2}{p-1} \binom{k}{2p-2} p^{2(k-2p+2)}. \quad (10)$$

Let  $\tilde{\mathcal{G}}(k, p)$  be the set of good closed walks  $w$  of length  $k$  on  $[p]$  where vertices first appear in  $w$  in the order  $1, 2, \dots, p$ . It is easy to check  $|\mathcal{G}(n, k, p)| = n(n-1) \cdots (n-p+1) |\tilde{\mathcal{G}}(k, p)|$ . The main contribution from Vu's paper [18] is the following improved bound (see [18], Lemma 4.1):

$$|\tilde{\mathcal{G}}(k, p)| \leq \binom{k}{2p-2} 2^{2k-2p+3} p^{k-2p+2} (k-2p+4)^{k-2p+2}. \quad (11)$$

We will use this bound to derive the following Lemma.

**Lemma 1** *For any even integer  $k$  such that  $k^4 \leq \frac{\Delta}{32}$ , we have*

$$|\mathbb{E}(\text{Trace}(B^k))| \leq 2^{k+2} n \Delta^{k/2}. \quad (12)$$

**Proof:** Let  $[n]^{\underline{p}} := \{(v_1, v_2, \dots, v_p) \in [n]^p : v_1, v_2, \dots, v_p \text{ are distinct}\}$ . Define

$$\phi: \mathcal{G}(n, k, p) \rightarrow \tilde{\mathcal{G}}(k, p) \times [n]^{\underline{p}}$$

as follows. For a good closed walk  $w = i_1 i_2 \dots i_k i_1 \in \mathcal{G}(n, k, p)$ , let  $v_1, v_2, \dots, v_p$  be the list of  $p$  vertices in the order as they appear in  $w$ . Replacing  $v_i$  by  $i$  for  $1 \leq i \leq p$ , we get a good closed walk  $\tilde{w} \in \tilde{\mathcal{G}}(k, p)$ . Now we define  $\phi(w) = (\tilde{w}, (v_1, v_2, \dots, v_p))$ . Clearly  $\phi$  is a bijection.

For any  $w \in \tilde{\mathcal{G}}(k, p)$ , we define a rooted tree  $T$  (with root 1) on the vertex set  $[p]$  as follows:

$$i_j i_{j+1} \in E(T) \text{ if and only if } i_{j+1} \notin \{i_1, i_2, \dots, i_j\}.$$

Equivalently, the edge  $i_j i_{j+1} \in E(T)$  if it brings in a new vertex when it occurs first time. For  $2 \leq l \leq p$ , let  $\eta(l)$  be the parent of  $l$ . Since  $\sigma_e^2 \leq K^2 = 1$  always holds, we can discard

those terms  $\sigma_e^2$  for  $e \notin T(w)$ . We get

$$\begin{aligned}
\sum_{w \in \mathcal{G}(n, k, p)} \prod_{e \in E(w)} \sigma_e^2 &= \sum_{\tilde{w} \in \tilde{\mathcal{G}}(k, p)} \sum_{(v_1, \dots, v_p) \in [n]^p} \prod_{xy \in E(\tilde{w})} \sigma_{v_x v_y}^2 \\
&\leq \sum_{\tilde{w} \in \tilde{\mathcal{G}}(k, p)} \sum_{v_1=1}^n \sum_{v_2=1}^n \cdots \sum_{v_p=1}^n \prod_{xy \in E(T)} \sigma_{v_x v_y}^2 \\
&= \sum_{\tilde{w} \in \tilde{\mathcal{G}}(k, p)} \sum_{v_1=1}^n \sum_{v_2=1}^n \cdots \sum_{v_p=1}^n \prod_{y=2}^p \sigma_{v_{\eta(y)} v_y}^2 \\
&= \sum_{\tilde{w} \in \tilde{\mathcal{G}}(k, p)} \sum_{v_1=1}^n \sum_{v_2=1}^n \cdots \sum_{v_{p-1}=1}^n \prod_{y=2}^{p-1} \sigma_{v_{\eta(y)} v_y}^2 \sum_{v_p=1}^n \sigma_{v_{\eta(p)} v_p}^2 \\
&\leq \Delta \sum_{\tilde{w} \in \tilde{\mathcal{G}}(k, p)} \sum_{v_1=1}^n \sum_{v_2=1}^n \cdots \sum_{v_{p-1}=1}^n \prod_{y=2}^{p-1} \sigma_{v_{\eta(y)} v_y}^2 \\
&\leq \cdots \\
&\leq \Delta^{p-1} \sum_{\tilde{w} \in \tilde{\mathcal{G}}(k, p)} \sum_{v_1=1}^n 1. \\
&= n \Delta^{p-1} |\tilde{\mathcal{G}}(k, p)|.
\end{aligned}$$

Combining it with inequality (9), we get

$$\begin{aligned}
|\mathbb{E}(\text{Trace}(B^k))| &\leq \sum_{w \in \mathcal{G}(n, k)} \prod_{e \in E(w)} \sigma_e^2 \\
&= \sum_{p=2}^{k/2+1} \sum_{w \in \mathcal{G}(n, k)} \prod_{e \in E(w)} \sigma_e^2 \\
&\leq \sum_{p=2}^{k/2+1} n \Delta^{p-1} |\tilde{\mathcal{G}}(k, p)| \\
&\leq n \sum_{p=2}^{k/2+1} \Delta^{p-1} \binom{k}{2p-2} 2^{2k-2p+3} p^{k-2p+2} (k-2p+4)^{k-2p+2}.
\end{aligned}$$

In the last step, we applied Vu's bound (11). Let  $S(n, k, p) := n \Delta^{p-1} \binom{k}{2p-2} 2^{2k-2p+3} p^{k-2p+2} (k-2p+4)^{k-2p+2}$ . One can show

$$S(n, k, p-1) \leq \frac{16k^4}{\Delta} S(n, k, p).$$

When  $k^4 \leq \frac{\Delta}{32}$ , we have  $S(n, k, p-1) \leq \frac{1}{2}S(n, k, p)$ . Thus,

$$\begin{aligned}
|\mathbb{E}(\text{Trace}(B^k))| &\leq \sum_{p=2}^{k/2+1} S(n, k, p) \\
&\leq S(n, k, k/2+1) \sum_{p=2}^{k/2+1} \left(\frac{1}{2}\right)^{k/2+1-p} \\
&< 2S(n, k, k/2+1) \\
&= n2^{k+2}\Delta^{k/2}.
\end{aligned}$$

The proof of this Lemma is finished.  $\square$

Now we are ready to prove Theorem 6.

**Proof of Theorem 6:** We have

$$\begin{aligned}
\Pr(\|B\| \geq 2\sqrt{\Delta} + C\Delta^{1/4} \ln n) &= \Pr(\|B\|^k \geq (2\sqrt{\Delta} + C\Delta^{1/4} \ln n)^k) \\
&\leq \Pr(\text{Trace}(\|B\|^k) \geq (2\sqrt{\Delta} + C\Delta^{1/4} \ln n)^k) \\
&\leq \frac{\mathbb{E}(\text{Trace}(\|B\|^k))}{(2\sqrt{\Delta} + C\Delta^{1/4} \ln n)^k} \\
&\leq \frac{n2^{k+2}\Delta^{k/2}}{(2\sqrt{\Delta} + C\Delta^{1/4} \ln n)^k} \\
&= 4ne^{-(1+o(1))\frac{C}{2}k\Delta^{-1/4} \ln n}.
\end{aligned}$$

Setting  $k = \left(\frac{\Delta}{32}\right)^{1/4}$ , we get

$$\Pr(\|B\| \geq 2\sqrt{\Delta} + C\Delta^{1/4} \ln n) = o(1)$$

for sufficiently large  $C$ . The proof of Theorem 6 is finished.  $\square$

**Proof of Theorem 1:** Let  $B = A - \bar{A}$ . Notice  $|b_{ij}| \leq 1$  and  $\text{Var}(b_{ij}) = p_{ij}(1 - p_{ij}) \leq p_{ij}$ .

Apply Theorem 6 to  $B$  with  $K = 1$ ,  $\sigma_{ij}^2 = p_{ij}$ , and  $\Delta = \max_{1 \leq i \leq 1} \sum_{j=1}^n p_{ij}$ . We get

$$\|B\| \leq 2\sqrt{\Delta} + C\Delta^{1/4} \ln n.$$

When  $\Delta \gg \ln^4 n$ , we have

$$\|B\| \leq (2 + o(1))\sqrt{\Delta}.$$

Applying Weyl's Theorem, we get

$$|\lambda_i(A) - \lambda_i(\bar{A})| \leq \|A - \bar{A}\| \leq (2 + o(1))\sqrt{\Delta}.$$

The proof of Theorem 1 is completed.  $\square$

### 3 Lemmas for Laplacian eigenvalues

In this section, we will present some necessary lemmas for proving Theorem 2. Recall  $G$  is an edge-independent random graph over  $[n]$  such that  $\{i, j\}$  forms as an edge with probability  $p_{ij}$  independently. For  $1 \leq i \leq j \leq n$ , let  $X_{ij}$  be the random indicator variable for  $\{i, j\}$  being an edge; we have  $\Pr(X_{ij} = 1) = p_{ij}$  and  $\Pr(X_{ij} = 0) = 1 - p_{ij}$ . For each vertex  $i \in [n]$ ,

we use  $d_i$  and  $t_i$  to denote the degree and the expected degree of vertex  $i$  in  $G$  respectively. We have

$$d_i = \sum_{j=1}^n X_{ij} \text{ and } t_i = \sum_{j=1}^n p_{ij}.$$

Let  $D$  (and  $T$ ) be the diagonal matrix with  $D_{ii} = d_i$  (and  $T_{ii} = t_i$ ) respectively. The matrix  $T$  is the expectation of  $D$ . Note that we use  $A$  and  $L$  to denote the adjacency matrix and the Laplacian matrix of  $G$ . Here  $L = I - D^{-1/2}AD^{-1/2}$ . Moreover, we let  $\bar{A} := E(A) = (p_{ij})_{i,j=1}^n$  be the expectation of  $A$ . We also define  $\bar{L} = I - T^{-1/2}\bar{A}T^{-1/2}$ . The matrix  $\bar{L}$  can be viewed as the “expected Laplacian matrix” of  $G$ .

For notational convenience, we write eigenvalues of the expected Laplacian matrix  $\bar{L}$  as  $\mu_1, \mu_2, \dots, \mu_n$  such that

$$|1 - \mu_1| \geq |1 - \mu_2| \geq \dots \geq |1 - \mu_n|.$$

By the definition of  $k$  and  $\Lambda$ , we have  $\Lambda = \{\mu_1, \dots, \mu_k\}$  and  $|1 - \mu_i| = o(1/\sqrt{\ln n})$  for all  $i \geq k+1$ .

For  $1 \leq i \leq n$ , let  $\phi^{(i)}$  be the orthonormal eigenvector of  $\mu_i$  for  $\bar{L}$ . Observe that  $\phi^{(i)}$  can also be viewed as the orthonormal eigenvector for  $T^{-1/2}\bar{A}T^{-1/2}$  corresponding to the eigenvalue  $1 - \mu_i$ . Thus we can rewrite  $T^{-1/2}\bar{A}T^{-1/2} = \sum_{i=1}^n (1 - \mu_i)\phi^{(i)}\phi^{(i)'}.$  Let  $M = \sum_{i=1}^k (1 - \mu_i)\phi^{(i)}\phi^{(i)'}$  and  $N = \sum_{i=k+1}^n (1 - \mu_i)\phi^{(i)}\phi^{(i)'}$ . Observe a fact  $\|N\| = o(1/\sqrt{\ln n})$  and this fact will be used later. For a square matrix  $B$ , we define

$$f(B) = D^{-1/2}T^{1/2}BT^{1/2}D^{-1/2} - B.$$

We shall rewrite  $L - \bar{L}$  as a sum of four matrices. Notice  $L - \bar{L} = D^{-1/2}AD^{-1/2} - T^{-1/2}\bar{A}T^{-1/2}$ . It is easy to verify  $L - \bar{L} = M_1 + M_2 + M_3 + M_4$ , where  $M_i$  are following.

$$\begin{aligned} M_1 &= T^{-1/2}(A - \bar{A})T^{-1/2}, \\ M_2 &= f(M_1), \\ M_3 &= f(N), \\ M_4 &= f(M). \end{aligned}$$

Here the matrices  $M$  and  $N$  are defined above. We will bound  $\|M_i\|$  for  $1 \leq i \leq 4$  separately.

**Lemma 2** *If  $\delta \gg \ln^4 n$ , then*

$$\|M_1\| \leq (2 + o(1))\frac{1}{\sqrt{\delta}}.$$

**Proof:** We are going to apply Theorem 6 to  $B = M_1 = T^{-1/2}(A - \bar{A})T^{-1/2}$ . Note

$$|b_{ij}| \leq (t_i t_j)^{-1/2} \leq 1/\delta,$$

and

$$\text{Var}(b_{ij}) = \text{Var}((t_i t_j)^{-1/2}(a_{ij} - p_{ij})) = \frac{p_{ij}(1 - p_{ij})}{t_i t_j} \leq \frac{p_{ij}}{t_i t_j}.$$

Let  $K = 1/\delta$  and  $\sigma_{ij}^2 = \frac{p_{ij}}{t_i t_j}$ . We have

$$\Delta(B) = \max_{1 \leq i \leq n} \sum_{j=1}^n \frac{p_{ij}}{t_i t_j} \leq \max_{1 \leq i \leq n} \sum_{j=1}^n \frac{p_{ij}}{t_i \delta} = \frac{1}{\delta}.$$



By Theorem 6, we get

$$\|B\| \leq \frac{2}{\sqrt{\delta}} + C\delta^{-3/4} \ln n.$$

When  $\delta \gg \ln^4 n$ , we have  $C\delta^{-3/4} \ln n = o(\frac{2}{\sqrt{\delta}})$ . Thus, we get  $\|M_1\| \leq \frac{2+o(1)}{\sqrt{\delta}}$  and the proof of the lemma is finished.  $\square$

We have the following lemma on the function  $f$ .

**Lemma 3** *If  $\|B\| = o(1/\sqrt{\ln n})$ , then  $\|f(B)\| = o(1/\sqrt{\delta})$ .*

Before we prove Lemma 3, we have two corollaries.

**Corollary 2** *If  $\delta \gg \ln^4 n$ , then we have  $\|M_2\| = o(1/\sqrt{\delta})$ .*

**Proof:** Recall  $M_2 = f(M_1)$ . By Lemma 2, we have

$$\|M_1\| \leq \frac{2+o(1)}{\sqrt{\delta}} = o(1/\sqrt{\ln n}).$$

By Lemma 3, we have this Corollary.  $\square$

**Corollary 3** *If  $\delta \gg \ln^4 n$ , then we have  $\|M_3\| = o(1/\sqrt{\delta})$ .*

**Proof:** Recall  $M_3 = f(N)$ . By the definition of  $N$ , we have  $\|N\| = o(1/\sqrt{\ln n})$ . Lemma 3 gives us the Corollary.  $\square$

To prove Lemma 3, we need the following Chernoff inequality.

**Theorem 7** [2] *Let  $X_1, \dots, X_n$  be independent random variables with*

$$\Pr(X_i = 1) = p_i, \quad \Pr(X_i = 0) = 1 - p_i.$$

*We consider the sum  $X = \sum_{i=1}^n X_i$ , with expectation  $E(X) = \sum_{i=1}^n p_i$ . Then we have*

$$\begin{aligned} \text{(Lower tail)} \quad & \Pr(X \leq E(X) - \lambda) \leq e^{-\lambda^2/2E(X)}, \\ \text{(Upper tail)} \quad & \Pr(X \geq E(X) + \lambda) \leq e^{-\frac{\lambda^2}{2(E(X)+\lambda/3)}}. \end{aligned}$$

We can use the lemma above to prove the degree of each vertex concentrates around its expectation.

**Lemma 4** *Assume  $t_i \geq \ln n$  for  $1 \leq i \leq n$ . Then with probability at least  $1 - \frac{1}{n^2}$ , for all  $1 \leq i \leq n$  we have*

$$|d_i - t_i| \leq 3\sqrt{t_i \ln n}.$$

**Proof:** Recall  $X_{ij}$  is the random indicator variable for  $\{i, j\}$  being an edge. Note  $d_i = \sum_{j=1}^n X_{ij}$  and  $E(d_i) = \sum_{j=1}^n p_{ij} = t_i$ . Applying the lower tail of Chernoff's inequality with  $\lambda = 3\sqrt{t_i \ln n}$ , we have

$$\Pr(d_i - t_i \leq -\lambda) \leq e^{-\lambda^2/2t_i} = \frac{1}{n^{9/2}}.$$

Applying the upper tail of Chernoff's inequality with  $\lambda = 3\sqrt{t_i \ln n}$ , we have

$$\Pr(d_i - t_i \geq \lambda) \leq e^{-\frac{\lambda^2}{2(t_i+\lambda/3)}} \leq \frac{1}{n^{27/8}}.$$

The union bound gives the lemma.  $\square$

By Lemma 4, we can write  $d_i = (1 + o(1))t_i$  for  $1 \leq i \leq n$ .

**Lemma 5** When  $\delta \gg \ln n$ , we have

$$\|D^{-1/2}T^{1/2} - I\| = O\left(\sqrt{\frac{\ln n}{\delta}}\right) \text{ and } \|T^{1/2}D^{-1/2}\| = 1 + o(1).$$

**Proof:** We note that  $D^{-1/2}T^{1/2} - I$  is diagonal and the  $(i, i)$ -the entry is  $\sqrt{t_i/d_i} - 1$ . We have

$$\begin{aligned} \left| \frac{\sqrt{t_i}}{\sqrt{d_i}} - 1 \right| &= \left| \frac{t_i - d_i}{\sqrt{d_i}(\sqrt{t_i} + \sqrt{d_i})} \right| \\ &\leq \left( \frac{3}{2} + o(1) \right) \sqrt{\frac{\ln n}{t_i}} \\ &= O\left(\sqrt{\frac{\ln n}{\delta}}\right). \end{aligned}$$

The first part of this lemma is proved while the second part follows from the triangle inequality. The proof of the lemma is finished.  $\square$

We are ready to prove Lemma 3.

**Proof of Lemma 3:** Recall that  $f(B) = D^{-1/2}T^{1/2}BT^{1/2}D^{-1/2} - B$ . We have

$$\begin{aligned} f(B) &= D^{-1/2}T^{1/2}BT^{1/2}D^{-1/2} - B \\ &= D^{-1/2}T^{1/2}BT^{1/2}D^{-1/2} - BT^{1/2}D^{-1/2} + BT^{1/2}D^{-1/2} - B \\ &= (D^{-1/2}T^{1/2} - I)BT^{1/2}D^{-1/2} + B(T^{1/2}D^{-1/2} - I). \end{aligned}$$

Recall Lemma 5. By the triangle inequality, we have

$$\begin{aligned} \|f(B)\| &\leq \|D^{-1/2}T^{1/2} - I\| \|B\| \|T^{1/2}D^{-1/2}\| + \|B\| \|(T^{1/2}D^{-1/2} - I)\| \\ &\leq O\left(\sqrt{\frac{\ln n}{\delta}}\right) \|B\| (1 + o(1)) + \|B\| O\left(\sqrt{\frac{\ln n}{\delta}}\right) \\ &= o\left(\frac{1}{\sqrt{\delta}}\right). \end{aligned}$$

We use the assumption  $\|B\| = o(1/\sqrt{\ln n})$  in the last step and we completed the proof of the lemma.  $\square$

## 4 Proof of Theorem 2

It remains to estimate  $\|M_4\|$ . Recall  $M_4 = f(M)$  and  $M = \sum_{i=1}^k (1 - \mu_i) \phi^{(i)} \phi^{(i)'}.$

For  $1 \leq i \leq n$ , write  $\phi^{(i)}$  as a vector  $(\phi_1^{(i)}, \phi_2^{(i)}, \dots, \phi_n^{(i)})'$ . Let  $\|\phi^{(i)}\|_\infty$  be the maximum over  $\{|\phi_1^{(i)}|, |\phi_2^{(i)}|, \dots, |\phi_n^{(i)}|\}$ . We have the following lemma.

**Lemma 6** For each  $1 \leq i \leq n$ , we have

$$|1 - \mu_i| \cdot \|\phi^{(i)}\|_\infty \leq \frac{1}{\sqrt{\delta}}.$$

**Proof:** Assume  $\|\phi^{(i)}\|_\infty = |\phi_j^{(i)}|$  for some index  $j$ . Since  $\phi^{(i)}$  is the orthonormal eigenvector associated with  $1 - \mu_i$  for  $T^{-1/2}\bar{A}T^{-1/2}$ , we have  $T^{-1/2}\bar{A}T^{-1/2}\phi^{(i)} = (1 - \mu_i)\phi^{(i)}$ . In particular,  $(T^{-1/2}\bar{A}T^{-1/2}\phi^{(i)})_j = (1 - \mu_i)\phi_j^{(i)}$  holds. We have

$$\begin{aligned} |1 - \mu_i| \cdot \|\phi^{(i)}\|_\infty &= |1 - \mu_i| |\phi_j^{(i)}| \\ &= |(T^{-1/2}\bar{A}T^{-1/2}\phi^{(i)})_j| \\ &\leq \sum_{l=1}^n \frac{p_{jl} |\phi_l^{(i)}|}{\sqrt{t_j t_l}} \\ &\leq \left( \sum_{l=1}^n (\phi_l^{(i)})^2 \right)^{1/2} \left( \sum_{l=1}^n \frac{p_{jl}^2}{t_j t_l} \right)^{1/2} \\ &\leq \frac{1}{\sqrt{\delta}} \left( \sum_{l=1}^n \frac{p_{jl}}{t_j} \right)^{1/2} \\ &= \frac{1}{\sqrt{\delta}}. \end{aligned}$$

The lemma is proved.  $\square$

**Lemma 7** Assume  $\delta \gg \ln n$ . For  $1 \leq i \leq n$ , consider a random variable  $X_i := \frac{(d_i - t_i)^2}{t_i}$ . We have  $E(X_i) \leq 1$  and  $\text{Var}(X_i) \leq 2 + o(1)$  for  $1 \leq i \leq n$ , and  $\text{Cov}(X_i, X_j) = \frac{p_{ij}(1-p_{ij})(1-2p_{ij})}{t_i t_j}$  for  $1 \leq i \neq j \leq n$ .

**Proof:** For  $1 \leq i < j \leq n$ , recall that  $X_{ij}$  is the random indicator variable for  $\{i, j\}$  being an edge. We define  $Y_{ij} = X_{ij} - p_{ij}$ . Thus we have  $d_i - t_i = \sum_{j=1}^n Y_{ij}$ . Note that  $E(Y_{ij}) = 0$  and  $\text{Var}(Y_{ij}) = p_{ij}(1 - p_{ij})$ . We get  $E(X_i) = \frac{1}{t_i} \sum_{j=1}^n p_{ij}(1 - p_{ij}) \leq 1$ . We have

$$E(X_i^2) = \frac{1}{t_i^2} E \left( \sum_{j_1, j_2, j_3, j_4} Y_{ij_1} Y_{ij_2} Y_{ij_3} Y_{ij_4} \right).$$

Since we have  $E(Y_{ij}) = 0$ , the non-zero term occurs either  $j_1 = j_2 = j_3 = j_4$ , or  $j_1 = j_2 \neq j_3 = j_4$ , or  $j_1 = j_3 \neq j_2 = j_4$ , or  $j_1 = j_4 \neq j_2 = j_3$ . The contribution from the first case is

$$\frac{1}{t_i^2} \sum_{j=1}^n E(Y_{ij}^4) = \frac{1}{t_i^2} \sum_{j=1}^n (1 - p_{ij})^4 p_{ij} + p_{ij}^4 (1 - p_{ij}) \leq \frac{1}{t_i^2} \sum_{j=1}^n p_{ij} = \frac{1}{t_i} = o(1)$$

as we assume  $\delta \gg \ln n$ . The contribution from the second case is

$$\frac{1}{t_i^2} \sum_{j_1 \neq j_2} E(Y_{ij_1}^2) E(Y_{ij_2}^2) = \frac{1}{t_i^2} \sum_{j_1 \neq j_2} p_{ij_1} p_{ij_2} (1 - p_{ij_1})(1 - p_{ij_2}).$$

The contribution from the third case and the forth case equal the contribution from the second case. Thus

$$E(X_i^2) = o(1) + \frac{3}{t_i^2} \sum_{j_1 \neq j_2} p_{ij_1} p_{ij_2} (1 - p_{ij_1})(1 - p_{ij_2}).$$

We have

$$\begin{aligned} \mathbb{E}(X_i)^2 &= \frac{1}{t_i^2} \sum_{j=1}^n p_{ij}^2 (1 - p_{ij})^2 + \frac{1}{t_i^2} \sum_{j_1 \neq j_2} p_{ij_1} p_{ij_2} (1 - p_{ij_1})(1 - p_{ij_2}) \\ &= o(1) + \frac{1}{t_i^2} \sum_{j_1 \neq j_2} p_{ij_1} p_{ij_2} (1 - p_{ij_1})(1 - p_{ij_2}). \end{aligned}$$

Therefore we get

$$\begin{aligned} \text{Var}(X_i) &= \mathbb{E}(X_i^2) - \mathbb{E}(X_i)^2 \\ &= \frac{2}{t_i^2} \sum_{j_1 \neq j_2} (p_{ij_1} p_{ij_2} (1 - p_{ij_1})(1 - p_{ij_2})) + o(1) \\ &\leq \frac{2}{t_i^2} \sum_{j_1 \neq j_2} (p_{ij_1} p_{ij_2}) + o(1) \\ &\leq 2 + o(1). \end{aligned}$$

The covariance  $\text{Cov}(X_i, X_j)$  can be computed similarly. Here we omit the details.  $\square$

**Lemma 8** For any non-negative numbers  $a_1, a_2, \dots, a_n$ , let  $X = \sum_{i=1}^n a_i \frac{(d_i - t_i)^2}{t_i}$ . Set  $a = \max_{1 \leq i \leq n} \{a_i\}$ . Then we have

$$\Pr \left( X \geq \sum_{i=1}^n a_i + \eta \sqrt{a \sum_{i=1}^n a_i} \right) \leq \frac{2 + o(1)}{\eta^2}.$$

**Proof:** We have  $X = \sum_{i=1}^n a_i X_i$ , where  $X_i = \frac{(d_i - t_i)^2}{t_i}$ . We shall use the second moment method to show that  $X$  concentrates around its expectation. By Lemma 7, we have  $\mathbb{E}(X_i) \leq 1$ . Thus  $\mathbb{E}(X) \leq \sum_{i=1}^n a_i$ . For the variance, by Lemma 7, we obtain

$$\begin{aligned} \text{Var}(X) &= \sum_{i=1}^n a_i^2 \text{Var}(X_i) + \sum_{i \neq j} a_i a_j \text{Cov}(X_i, X_j) \\ &\leq \sum_{i=1}^n a_i^2 (2 + o(1)) + \sum_{i=1}^n \sum_{j=1}^n \frac{a_i a_j p_{ij} (1 - p_{ij})(1 - 2p_{ij})}{t_i t_j} \\ &\leq (2 + o(1))a \sum_{i=1}^n a_i + \frac{a}{\delta} \sum_{i=1}^n a_i \sum_{j=1}^n \frac{p_{ij}}{t_i} \\ &= (2 + o(1))a \sum_{i=1}^n a_i. \end{aligned}$$

Applying Chebychev's inequality, we have

$$\begin{aligned} \Pr \left( X \geq \sum_{i=1}^n a_i + \eta \sqrt{a \sum_{i=1}^n a_i} \right) &\leq \Pr \left( X - \mathbb{E}(X) \geq \eta \sqrt{a \sum_{i=1}^n a_i} \right) \\ &\leq \frac{\text{Var}(X)}{\eta^2 a \sum_{i=1}^n a_i} \\ &\leq \frac{2 + o(1)}{\eta^2}. \end{aligned}$$

□

We are ready to prove an upper bound on  $\|M_4\|$ .

**Lemma 9** *If  $\delta \gg \{k, \ln^4 n\}$ , then we have*

$$\|M_4\| \leq (1 + o(1)) \frac{\sqrt{\sum_{\lambda \in \Lambda} (1 - \lambda)^2}}{\sqrt{\delta}}.$$

**Proof:** Let  $\Phi := (\phi^{(1)}, \dots, \phi^{(k)})$  be an  $n \times k$  matrix such that its columns are mutually orthogonal and  $Q$  be a diagonal  $k \times k$  matrix such that  $Q_{ii} = 1 - \mu_i$ . We have  $M = \sum_{i=1}^k (1 - \mu_i) \phi^{(i)} \phi^{(i)'} = \Phi Q \Phi'$ . Thus,

$$\begin{aligned} M_4 &= D^{-1/2} T^{1/2} M T^{1/2} D^{-1/2} - M \\ &= D^{-1/2} T^{1/2} M T^{1/2} D^{-1/2} - M T^{1/2} D^{-1/2} + M T^{1/2} D^{-1/2} - M \\ &= (D^{-1/2} T^{1/2} - I) M T^{1/2} D^{-1/2} + M (T^{1/2} D^{-1/2} - I) \\ &= (D^{-1/2} T^{1/2} - I) \Phi Q \Phi' T^{1/2} D^{-1/2} + \Phi Q \Phi' (T^{1/2} D^{-1/2} - I). \end{aligned}$$

Let  $U = (D^{-1/2} T^{1/2} - I) \Phi Q$ . By the definition of  $\Phi$ , we have  $\|\Phi\| = 1$ . Since  $\|T^{1/2} D^{-1/2}\| = 1 + o(1)$ , by the triangle inequality, we get

$$\begin{aligned} \|M_4\| &= \|U \Phi' T^{1/2} D^{-1/2} + \Phi U'\| \\ &\leq \|U\| \|\Phi'\| \|T^{1/2} D^{-1/2}\| + \|\Phi\| \|U'\| \\ &= (2 + o(1)) \|U\|. \end{aligned}$$

By the definition of the norm of a non-square matrix, we have

$$\begin{aligned} \|U\| &= \sqrt{\|U U'\|} \\ &\leq \sqrt{\text{Trace}(U U')} \\ &= \sqrt{\text{Trace}((D^{-1/2} T^{1/2} - I) \Phi Q Q \Phi' (T^{1/2} D^{-1/2} - I))} \\ &= \sqrt{\sum_{j=1}^n \sum_{i=1}^k (1 - \mu_i)^2 \left(\phi_j^{(i)}\right)^2 \left(\frac{\sqrt{t_j}}{\sqrt{d_j}} - 1\right)^2}. \end{aligned}$$

Let  $a_j := \sum_{i=1}^k (1 - \mu_i)^2 \left(\phi_j^{(i)}\right)^2$ . We have the following estimate on the norm of  $U$ ,

$$\begin{aligned} \|U\|^2 &\leq \sum_{j=1}^n a_j \left(\frac{\sqrt{t_j}}{\sqrt{d_j}} - 1\right)^2 \\ &= \sum_{j=1}^n a_j \frac{(t_j - d_j)^2}{d_j (\sqrt{t_j} + \sqrt{d_j})^2} \\ &= (1 + o(1)) \sum_{j=1}^n a_j \frac{(t_j - d_j)^2}{4t_j^2} \\ &\leq \frac{1 + o(1)}{4\delta} \sum_{j=1}^n a_j \frac{(t_j - d_j)^2}{t_j}. \end{aligned}$$

Note that

$$\begin{aligned}
\sum_{j=1}^n a_j &= \sum_{j=1}^n \sum_{i=1}^k (1 - \mu_i)^2 \left( \phi_j^{(i)} \right)^2 \\
&= \sum_{i=1}^k (1 - \mu_i)^2 \sum_{j=1}^n \left( \phi_j^{(i)} \right)^2 \\
&= \sum_{i=1}^k (1 - \mu_i)^2.
\end{aligned}$$

By Lemma 6, we have

$$|1 - \mu_i| \cdot \|\phi^{(i)}\|_\infty \leq \frac{1}{\sqrt{\delta}}.$$

Hence, for  $1 \leq j \leq n$ , we get

$$a_j = \sum_{i=1}^k (1 - \mu_i)^2 \left( \phi_j^{(i)} \right)^2 \leq \frac{k}{\delta}.$$

If we let  $a = \max_{1 \leq j \leq n} \{a_j\}$ , then we have  $a \leq \frac{k}{\delta}$ .

Choose  $\eta := \sqrt[3]{\delta}/\sqrt[3]{k}$ ; we have  $\eta \rightarrow \infty$  as  $n$  approaches the infinity. Applying Lemma 8, with probability  $1 - o(1)$ , we have

$$\begin{aligned}
\sum_{j=1}^n a_j \frac{(t_j - d_j)^2}{t_j} &\leq \sum_{j=1}^n a_j + \eta \sqrt{a \sum_{j=1}^n a_j} \\
&\leq \sum_{i=1}^k (1 - \mu_i)^2 + \frac{\sqrt[3]{\delta}}{\sqrt[3]{k}} \sqrt{\frac{k}{\delta} \sum_{i=1}^k (1 - \mu_i)^2} \\
&= \left( \sqrt{\sum_{i=1}^k (1 - \mu_i)^2} + o(1) \right)^2.
\end{aligned}$$

Therefore, we get the following upper bounds on  $\|U\|$  and  $\|M_4\|$ ;

$$\|U\| \leq (1 + o(1)) \frac{\sqrt{\sum_{i=1}^k (1 - \mu_i)^2}}{2\sqrt{\delta}},$$

and

$$\|M_4\| \leq (1 + o(1)) \frac{\sqrt{\sum_{i=1}^k (1 - \mu_i)^2}}{\sqrt{\delta}} = (1 + o(1)) \frac{\sqrt{\sum_{\lambda \in \Lambda} (1 - \lambda)^2}}{\sqrt{\delta}}.$$

We proved the lemma.  $\square$

**Proof of Theorem 2:** Recall  $L - \bar{L} = M_1 + M_2 + M_3 + M_4$ . By the triangle inequality, we have  $\|L - \bar{L}\| \leq \|M_1\| + \|M_2\| + \|M_3\| + \|M_4\|$ . Combining Lemma 2, Corollary 2, Corollary

3, and Lemma 9, we get

$$\begin{aligned}
\|L - \bar{L}\| &\leq \|M_1\| + \|M_2\| + \|M_3\| + \|M_4\| \\
&\leq (2 + o(1)) \frac{1}{\sqrt{\delta}} + o\left(\frac{1}{\sqrt{\delta}}\right) + o\left(\frac{1}{\sqrt{\delta}}\right) + (1 + o(1)) \frac{\sqrt{\sum_{i=1}^k (1 - \mu_i)^2}}{\sqrt{\delta}} \\
&= \left(2 + \sqrt{\sum_{i=1}^k (1 - \mu_i)^2} + o(1)\right) \frac{1}{\sqrt{\delta}}.
\end{aligned}$$

Finally we apply Weyl's Theorem. The proof of Theorem 2 is finished.  $\square$

## References

- [1] N. Alon, M. Krivelevich, and V. H. Vu, Concentration of eigenvalues of random matrices, *Israel Math. J.*, **131** (2002), 259–267.
- [2] H. Chernoff, A note on an inequality involving the normal distribution, *Ann. Probab.*, **9** (1981), 533–535.
- [3] F. Chung, *Spectral graph theory*, AMS publications, 1997.
- [4] F. Chung, L. Lu, and V. H. Vu, Eigenvalues of random power law graphs, *Ann. Comb.*, **7** (2003), 21–33.
- [5] F. Chung, L. Lu, and V. H. Vu, Spectra of random graphs with given expected degrees, *Proc. Natl. Acad. Sci. USA*, **100**(11) (2003), 6313–6318.
- [6] F. Chung and M. Radcliffe, On the spectra of general random graphs, *Electron. J. Combin.*, **18**(1) (2011), P215.
- [7] A. Coja-Oghlan, On the Laplacian eigenvalues of  $G(n, p)$ , *Combin. Probab. Comput.*, **16**(6) (2007), 923–946.
- [8] A. Coja-Oghlan and A. Lanka, The spectral gap of random graphs with given expected degrees, *Electron. J. Combin.*, **16** (2009), R138.
- [9] D. M. Cvetković, M. Doob, and H. Sachs, *Spectra of Graphs, Theory and Applications*, Academic Press, 1980.
- [10] X. Ding and T. Jiang, Spectral distributions of adjacency and Laplacian matrices of random graphs, *Ann. Appl. Probab.*, **20**(6) (2010), 2086–2117.
- [11] U. Feige and E. Ofek, Spectral techniques applied to sparse random graphs, *Random Structures Algorithms*, **27**(2) (2005), 251–275.
- [12] J. Friedman, J. Kahn, and E. Szemerédi, On the second eigenvalue in random regular graphs, in Proc. 21st ACM Symposium on Theory of Computing, Association for Computing Machinery, New York, 1989, 587–598.
- [13] J. Friedman, On the second eigenvalue and random walks in random  $d$ -regular graphs, *Combinatorica*, **11**(4) (1991), 331–362.

- [14] J. Friedman, *A Proof of Alon's Second Eigenvalue Conjecture and Related Problem*, Memoirs of the American Mathematical Society 2008, 100 pp.
- [15] Z. Füredi and J. Komlós. The eigenvalues of random symmetric matrices, *Combinatorica*, **1**(3) 1981, 233–241.
- [16] M. Krivelevich and B. Sudakov, The largest eigenvalue of sparse random graphs, *Combin. Probab. Comput.*, **12** (2003), 61–72.
- [17] R. Oliveira, Concentration of the adjacency matrix and of the Laplacian in random graphs with independent edges, <http://arxiv.org/abs/0911.0600>.
- [18] V. H. Vu, Spectral norm of random matrices, *Combinatorica*, **27** (6) (2007), 721–736.
- [19] E. P. Wigner, On the distribution of the roots of certain symmetric matrices, *Ann. Math.*, **67** (1958), 325–327.